Elementary Derivation of the Hoffman Singleton Graph

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Background

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Theorem (Moore bound)

A regular graph of degree d and diameter k has at most $1 + \sum_{i=0}^{k-1} d(d-1)^i$ vertices.

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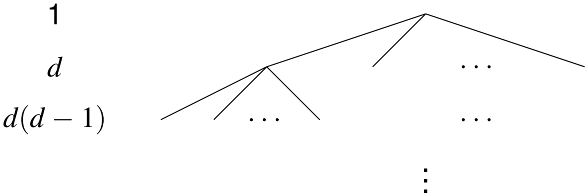

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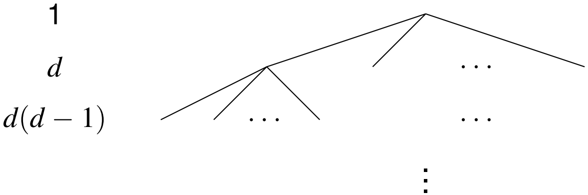

Question

Which graphs exist which achieve the Moore bound?

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A $(7, 2)$-Moore graph exists and is unique to isomorphism.

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If a $(57, 2)$-Moore graph exists then it is not vertex transitive.

Background (Further Results)

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There are no Moore graphs of diameter 4 or greater.

Theorem (Graham Higgman)

If a $(57, 2)$-Moore graph exists then it is not vertex transitive.

Theorem (Mačaj and Širáň)

The automorphism group of a $(57, 2)$-Moore graph contains at most 375 members.

What we aim to show

We will provide a different method of derivation of the Hoffman-Singleton graph. In this derivation we show:

- Uniqueness to isomorphism.
- Vertex transitivity.
- A construction of the graph.

To do this, we will only make deductions directly from the properties:

- Degree $d = 7$.
- Diameter $k = 2$.
- Girth $2k + 1 = 5$

The main idea we utilise is counting the 5-cycles via two different methods, which is the fundamental difference between this approach and a similar approach given by L. O. James.

Pentagons and Pentagrams

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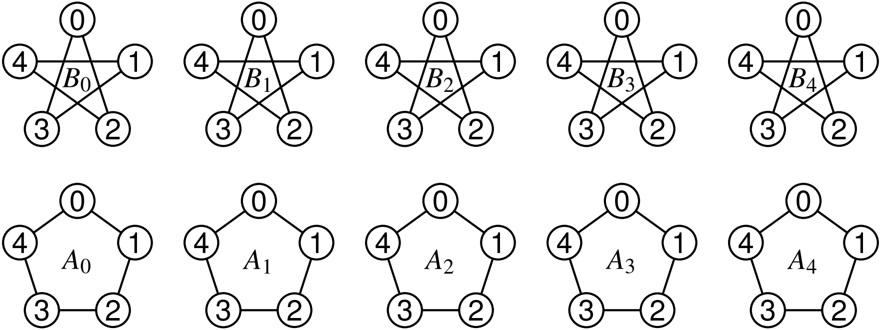

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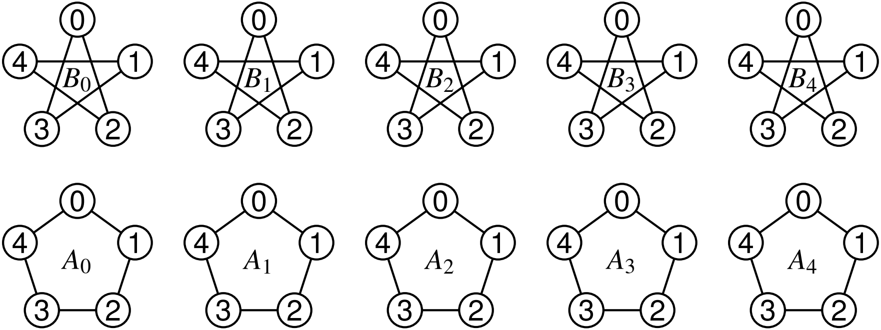

With the additional edges $a_{i,k} \sim b_{j,ij+k}$.

Counting 5-cycles

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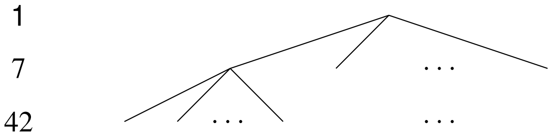

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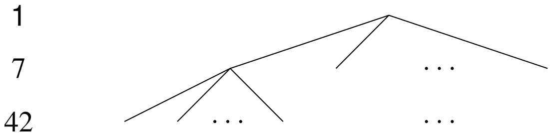

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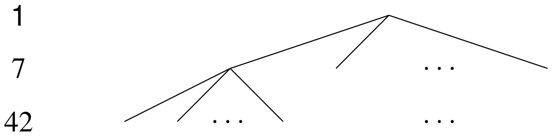

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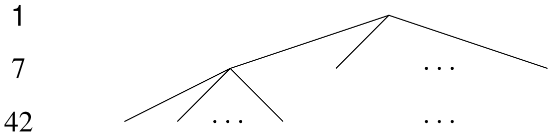
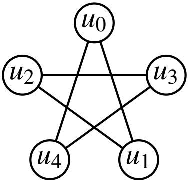
- There are 126 edges not in the tree.

Counting 5-cycles



- There is a bijection between edges not in the tree rooted at v and 5-cycles through v.
- There are 126 edges not in the tree.
- There are 1,260 5-cycles in the graph.

Derivation - (U, V, W)



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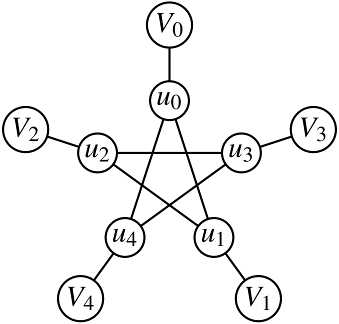

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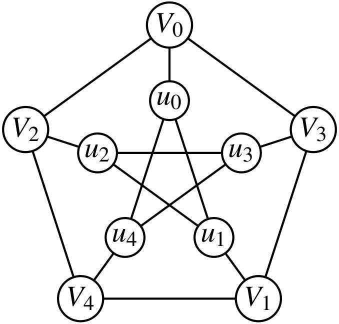

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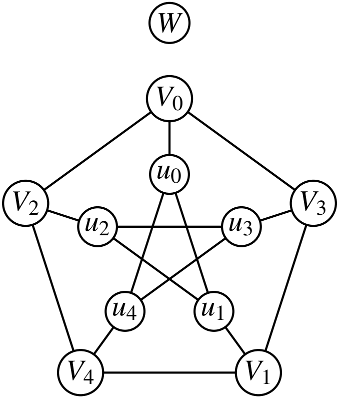

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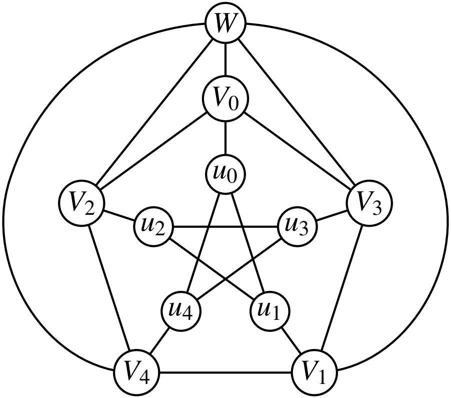
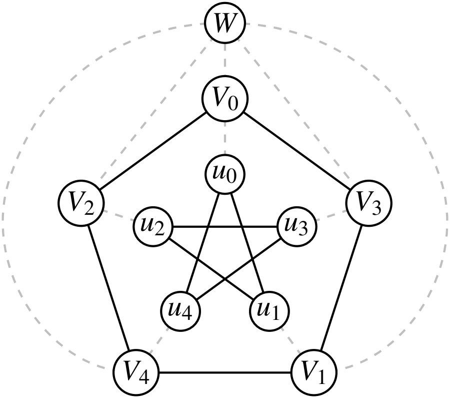
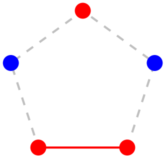


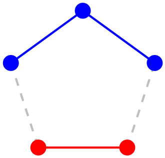
Derivation - ($A = U \cup W$, $B = V$)



Count 5-cycles

1 edge in E_A





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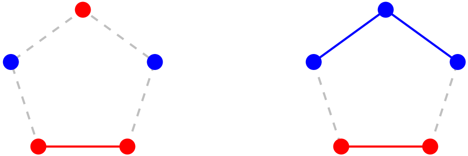
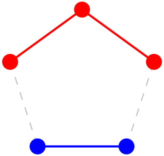


2 edges in E_A



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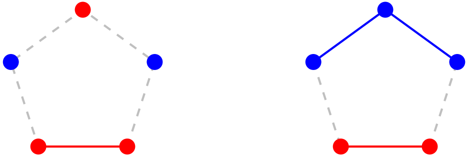

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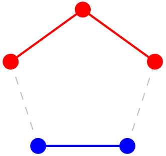
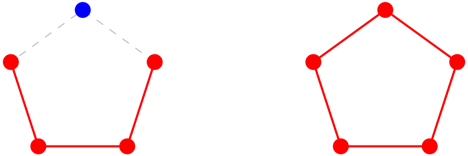


3+ edges in E_A



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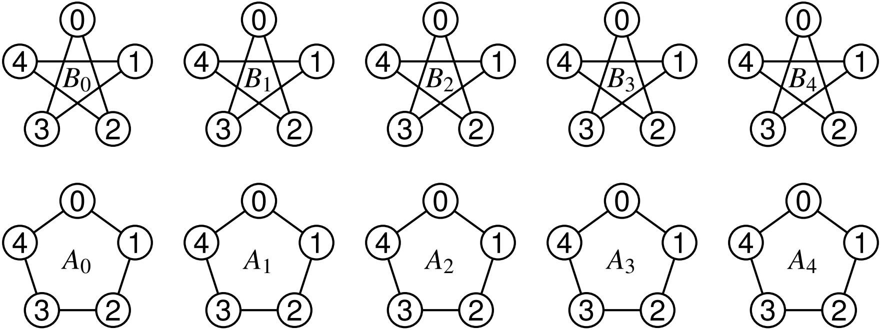

What is the value?

- Fails quickly in the $(57, 2)$-Moore graph case, and doesn't appear to give insight there.
- Perhaps it works due to the fact Moore graphs are extremal graphs for number of convex cycles (Azarija and Klavžar).
- ... Although, other graphs which fail to meet this bound will not permit the same counting argument.

The main value is the aesthetic value of a simpler argument to get a known result.

Thank you for listening!

Consistent Orientation