Automorphism Groups of the Gómez Graphs

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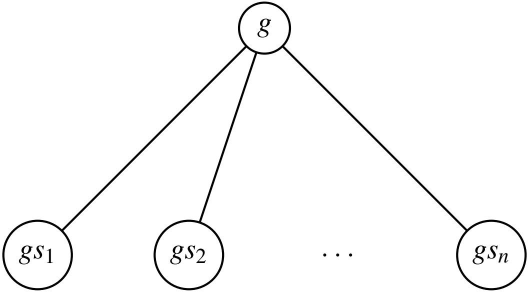
Quick Preliminary - What is a Cayley Graph?

Take a group G and a set S, define a graph $\Gamma(G, S)$ such that $V(\Gamma) = G$ and the neighbours of any $g \in V(G)$ are gs for each $s \in S$.



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- This family is similar to the Faber-Moore-Chen digraphs.
- We known for exactly which parameters these are Cayley, the argument essentially relies on determining their automorphism groups.

Definition

Choose some k and Δ, let $n = 2k + 1$, and fix a set B with $|B| = \Delta + k$. Define the Gómez graph $G(\Delta, k) = G$ such that:

$$V(G) = \{x_1 x_2 \ldots x_n : x_i \in B, x_i = x_j \Leftrightarrow i = j\}$$

$$x_1 x_2 \ldots x_n \to \begin{cases} x_2 x_3 \ldots x_n y, & y \in B \setminus \{x_i\} \\ x_2 x_3 \ldots x_n x_1, & \\ x_1 x_3 x_4 \ldots x_n x_2, & \\ \vdots & \\ x_2 x_3 \ldots x_k x_1 x_{k+2} \ldots x_n x_{k+1}. & \end{cases}$$

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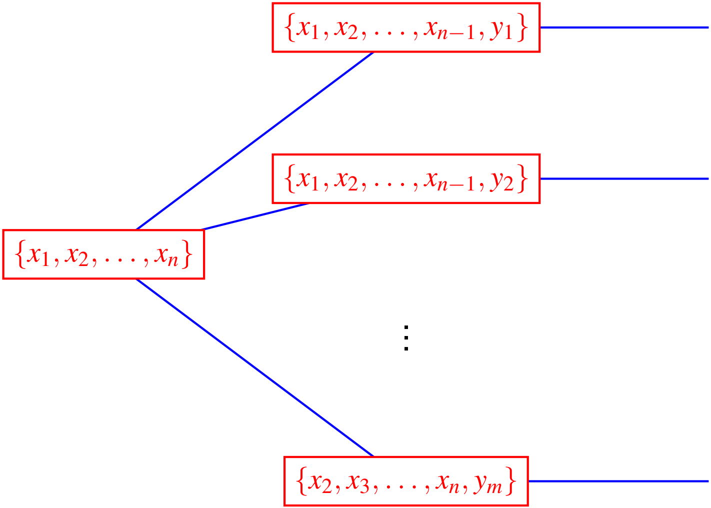
Strategy



$\{x_1, x_2, \ldots, x_{n-1}, y_1\}$
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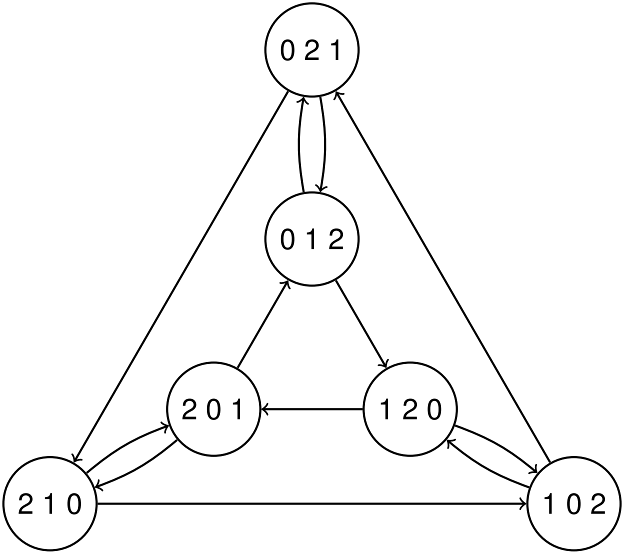
$\{x_1, x_2, \ldots, x_n\}$
$\vdots$
$\{x_2, x_3, \ldots, x_n, y_m\}$

$\{x_1, x_2, \ldots, x_n\}$-subgraph

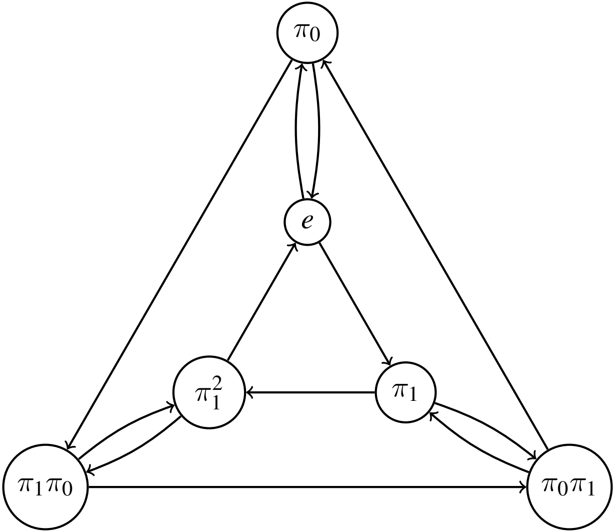
Rules for $k = 3$, $n = 2k + 1 = 7$.

n			
0	$\vert x_1 x_2 x_3 x_4 x_5 x_6 x_7$	$\rightarrow$	$x_2 x_3 x_4 x_5 x_6 x_7 x_1$
1	$x_1 \vert x_2 x_3 x_4 x_5 x_6 x_7$	$\rightarrow$	$x_1 x_3 x_4 x_5 x_6 x_7 x_2$
2	$x_1 x_2 \vert x_3 x_4 x_5 x_6 x_7$	$\rightarrow$	$x_2 x_1 x_4 x_5 x_6 x_7 x_3$
3	$x_1 x_2 x_3 \vert x_4 x_5 x_6 x_7$	$\rightarrow$	$x_2 x_3 x_1 x_5 x_6 x_7 x_4$

Example, $k = 1$



Example continued, $k = 1$ as a Cayley graph



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Hence, we can use automorphisms from the regular subset of $\mathrm{Aut}(\Gamma)$ to create an automorphism ϕ' with the property $\phi'(e) = e$ and $\phi'(\pi_i) \neq \pi_i$ for at least one i.

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Taking the hint from the paper of Gómez that G is n-reachable, we consider counting paths of length n starting from each vertex π_i and ending at e, as any automorphism of G must give a bijection between these sets of paths for a pair of vertices π_i, π_j.

Calculate some examples

n	3	5	7	9	11
π_0					
π_1					
π_2					
π_3					
π_4					
π_5					

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n	3	5	7	9	11
π_0	2				
π_1	1				
π_2					
π_3					
π_4					
π_5					

Calculate some examples

n	3	5	7	9	11
π_0	2	4			
π_1	1	2			
π_2		1			
π_3					
π_4					
π_5					

Calculate some examples

n	3	5	7	9	11
π_0	2	4	7		
π_1	1	2	4		
π_2		1	2		
π_3			1		
π_4					
π_5					

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n	3	5	7	9	11
π_0	2	4	7	11	
π_1	1	2	4	7	
π_2		1	2	4	
π_3			1	2	
π_4				1	
π_5					

Calculate some examples

n	3	5	7	9	11
π_0	2	4	7	11	16
π_1	1	2	4	7	11
π_2		1	2	4	7
π_3			1	2	4
π_4				1	2
π_5					1

The Search for These Paths

$k = 1$

$\pi_1 \rightarrow \pi_1\pi_0\pi_1\pi_0$

$\pi_0 \rightarrow \pi_0\pi_0\pi_0\pi_0, \pi_0\pi_1\pi_0\pi_1$

$k = 2$

$\pi_2 \rightarrow \pi_2\pi_1\pi_0\pi_2\pi_1\pi_0$

$\pi_1 \rightarrow \pi_1\pi_0\pi_0\pi_1\pi_0\pi_0, \pi_1\pi_0\pi_2\pi_1\pi_0\pi_2$

$\pi_0 \rightarrow \pi_0\pi_0\pi_0\pi_0\pi_0\pi_0, \pi_0\pi_0\pi_1\pi_0\pi_0\pi_1, \pi_0\pi_1\pi_0\pi_0\pi_1\pi_0, \pi_0\pi_2\pi_1\pi_0\pi_2\pi_1$

$k = 3$

$\pi_3 \rightarrow \pi_3\pi_2\pi_1\pi_0\pi_3\pi_2\pi_1\pi_0$

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Search Continued - and luck!

0

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1	0
0	0

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2	1	0
1	0	0
0	1	0
0	0	0

Search Continued - and luck!

3	2	1	0
2	1	0	0
1	0	1	0
1	0	0	0
0	2	1	0
0	1	0	0
0	0	1	0

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4	3	2	1	0
3	2	1	0	0
2	1	0	1	0
2	1	0	0	0
1	0	2	1	0
1	0	1	0	0
1	0	0	1	0
0	3	2	1	0
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5	4	3	2	1	0
4	3	2	1	0	0
3	2	1	0	1	0
3	2	1	0	0	0
2	1	0	2	1	0
2	1	0	1	0	0
2	1	0	0	1	0
1	0	3	2	1	0
1	0	2	1	0	0
1	0	1	0	1	0
1	0	0	2	1	0
0	4	3	2	1	0
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2	1	0	0	1	0
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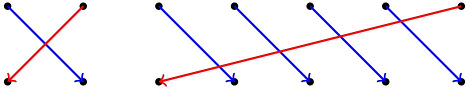
Example $n = 7, k = 3, i = 1$



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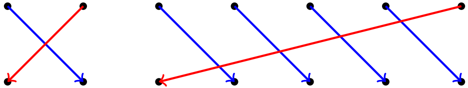

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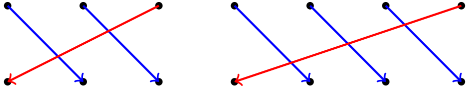

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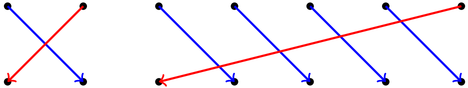

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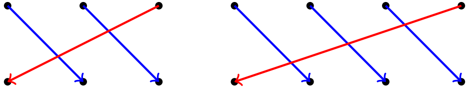
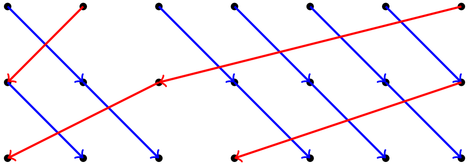
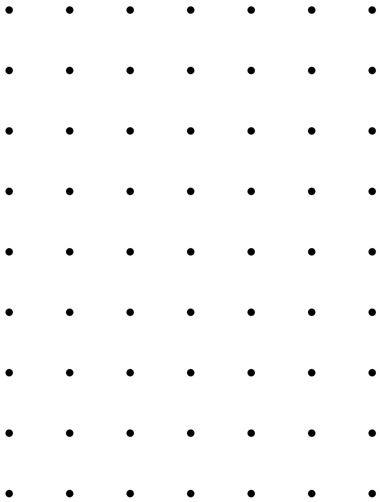


Composition



Proof of ii - Example $k = 3$



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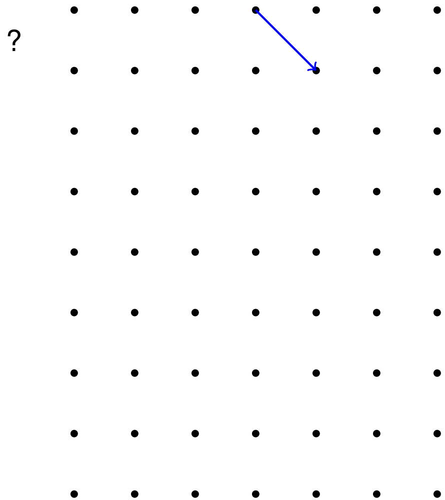

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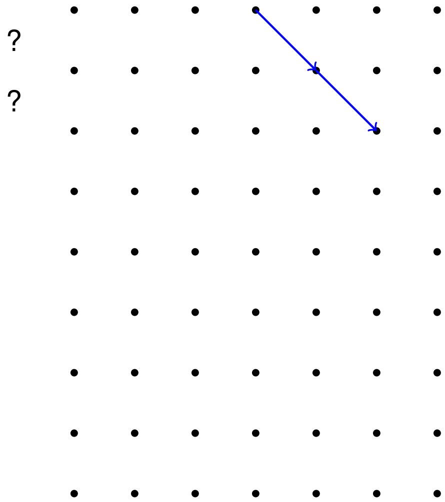

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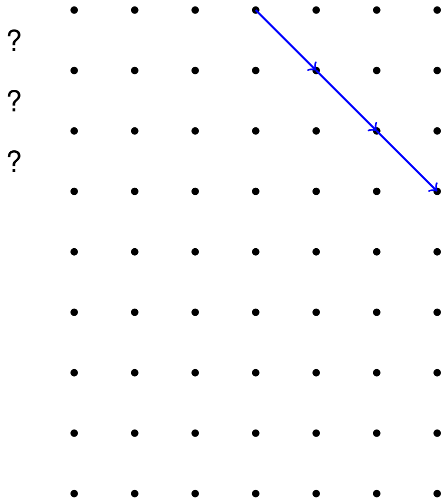

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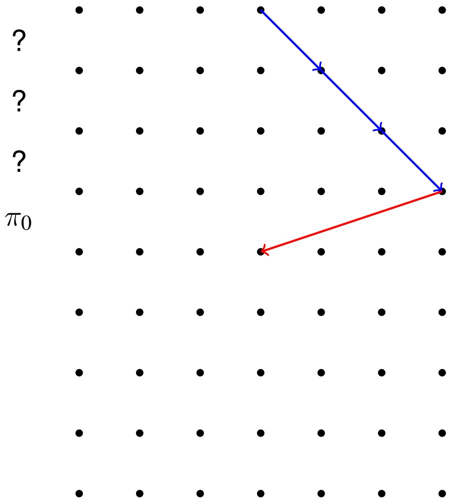

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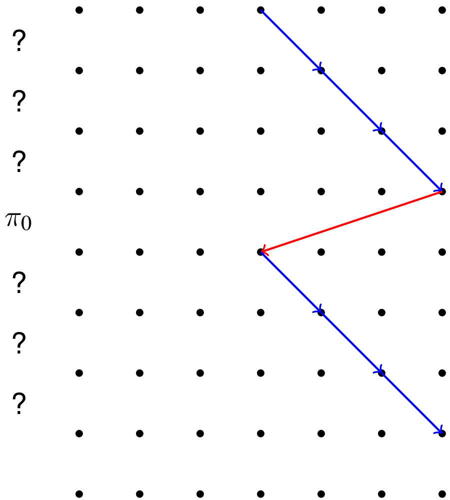

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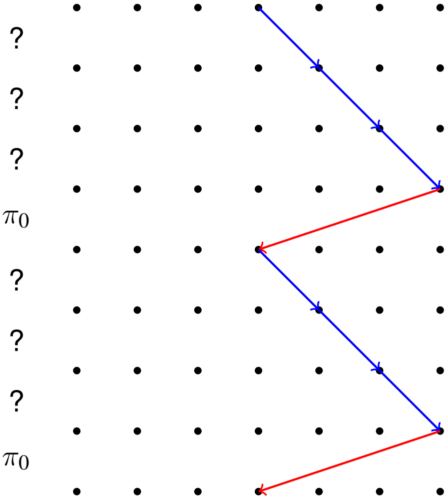

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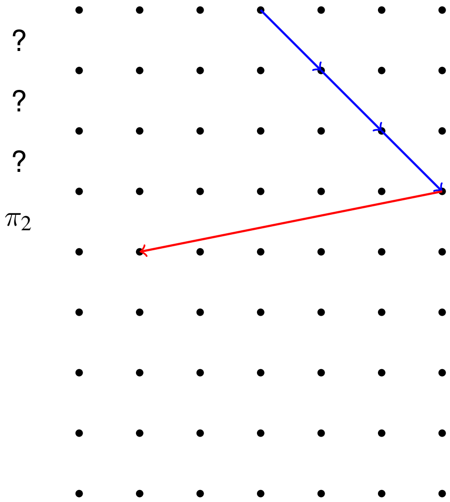

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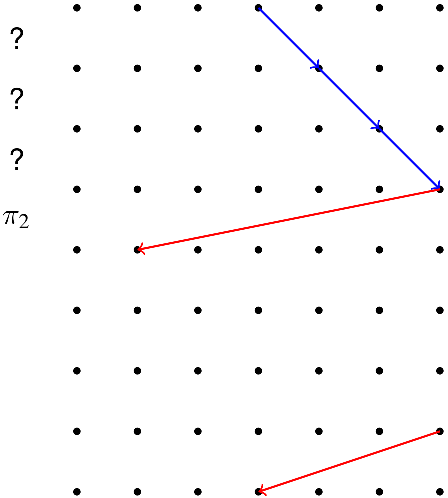

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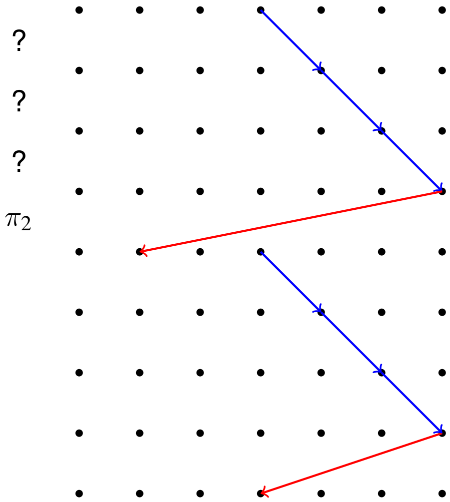

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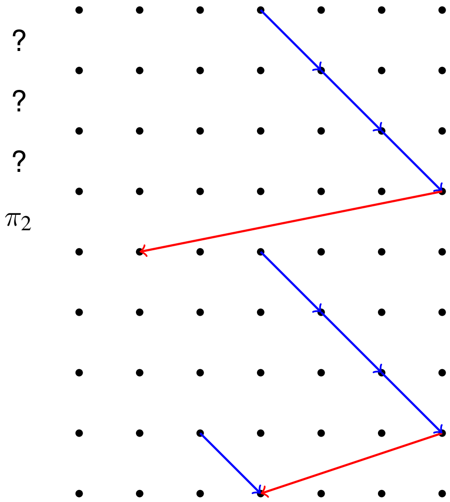

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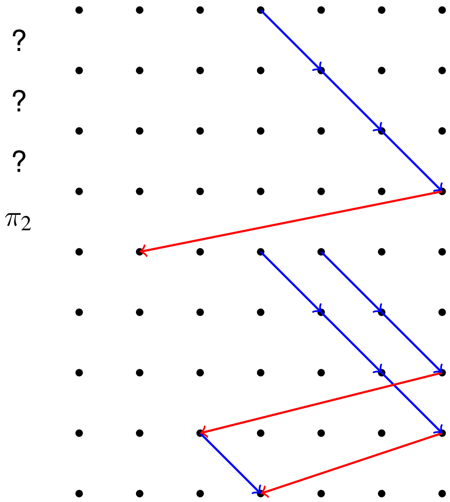
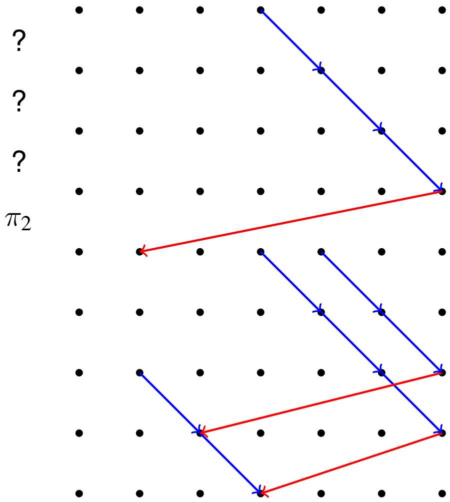
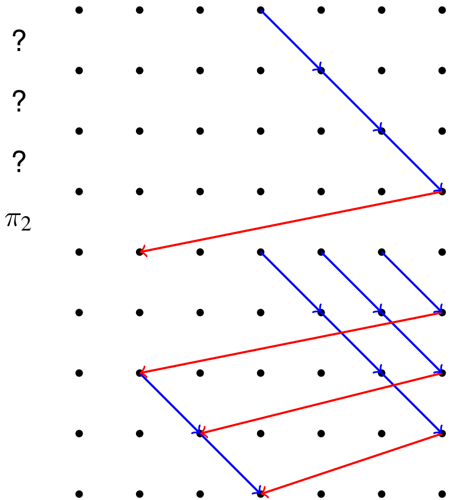


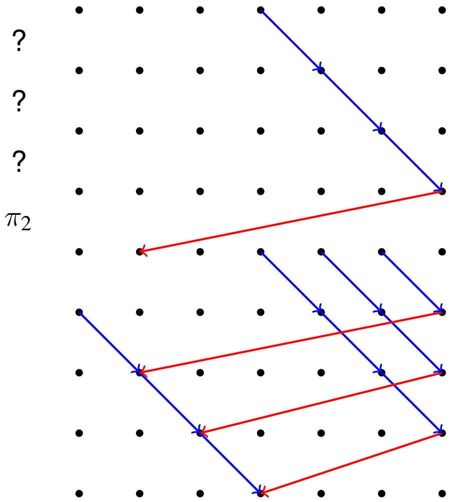
Proof of iii - Example $k = 3$, $\pi_i = \pi_2$



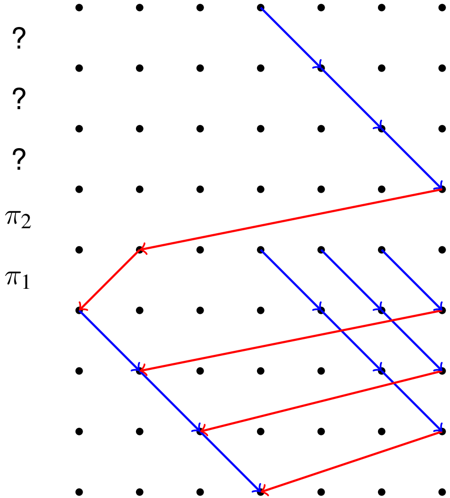
Proof of iii - Example $k = 3$, $\pi_i = \pi_2$



Proof of iii - Example $k = 3$, $\pi_i = \pi_2$



Proof of iii - Example $k = 3$, $\pi_i = \pi_2$



How to Complete From Here?

We now continue much in the same vein. Following the same mode of thinking, we can prove the structure of all paths of length $n+1$ from e back to itself, then complete the proof.
Subsequently, this can naturally be extended to an induction to settle the form of all automorphisms of the full Gómez graphs.

Thanks For Listening!