Elementary Derivation of the Hoffman-Singleton Graph

J. Fraser

November 7, 2018

Abstract

Graphs with diameter k and girth $2k + 1$ are known as Moore graphs. It can be shown that Moore graphs are necessarily regular graphs and, excluding trivial cases, cannot have a diameter of 3 or greater. Only three sets of parameters exist which may allow non-trivial Moore graphs, all of which are diameter 2. For degree 3 the unique Moore graph is the Petersen graph, for degree 7 the unique Moore graph is the Hoffman-Singleton graph, and for degree 57 it is unknown whether a Moore graph exists. In this paper we present a derivation of the Hoffman-Singleton graph as a Moore graph of degree 7 and diameter 2, that the Hoffman-Singleton graph is the unique to isomorphism Moore graph with these parameters, and further we pursue our method to characterise the automorphisms of the Hoffman-Singleton graph, showing that the graph is vertex transitive.

In graph theory we often study graphs which possess properties which make them in some sense extremal. Two such properties are the diameter and girth of a graph. The diameter of a graph is the longest distance between any two vertices in a graph, and the girth of a graph is the length of the shortest cycle within the graph. Supposing we have a graph with girth $2k + 1$, we see that the minimum value of the diameter is k, as otherwise consider two vertices in a cycle of length $2k + 1$ which are at distance k via edges in the cycle, as the graph is diameter less than k another path connects these vertices of length less than k, and hence these points lie in a cycle of length less than $2k + 1$, contradicting the assumed girth.

Hence, for a graph of diameter k we see the girth is at most $2k + 1$, and we may naturally ask what happens when the girth is exactly $2k + 1$. The problem of describing such graphs was originally proposed by E. F. Moore, and hence such graphs have become known as *Moore graphs*.

First, we see that such graphs trivially exist by considering either cycles of length $2k + 1$ or complete graphs. Considering these cases trivial, we search for other graphs which have this property. Before we continue further in our discussion we note that Moore graphs are *regular* (each vertex is incident to the same number of edges), thus inspiring the notation $(d,\ k)$-Moore graph to denote a Moore graph of diameter k and degree d. The interested reader may find a proof of this fact in [1].

The first paper to address the question of classifying Moore graphs was the paper [2] in 1960 by Hoffman and Singleton. They showed the non-existence of Moore graphs of diameter 3, that the only possible degrees for diameter 2 Moore graphs are 3, 7 and 57, and that the $(7,\ 2)$-Moore graph is unique to isomorphism. Subsequently the paper [3] of Ito and Bannai extended the argument used by Hoffman and Singleton to show the non-existence of Moore graphs of diameter 4 or greater. An alternative proof of this fact is also available in [5] from R. M. Damerell.

Hence, there are only three possible sets of parameters for non-trivial Moore graphs. Two of which are known to correspond to unique, vertex transitive graphs, namely the Petersen and Hoffman-Singleton graphs. The case of a possible $(57,\ 2)$-Moore graph is open, and is now one of the most famous open problems in graph theory, having persisted, at the time of writing, for some 57 years. Despite being unable to decide whether a $(57,\ 2)$-Moore graph exists, it was shown by Graham Higgman in a series of lectures that if such a graph exists it is not vertex transitive, in contrast to the only other known non-trivial Moore graphs. This proof is available in [6].

In this paper, we give a different approach to that used by Hoffman and Singleton to derive a $(7,\ 2)$-Moore graph, prove its uniqueness, and prove its vertex transitivity. The proof given is similar to that in [8], though discovered independently. One can trivially modify the approach used to the case of the $(3,\ 2)$-Moore graph also, and quickly derive the same facts of vertex transitivity and uniqueness.

We formally state the results we aim to prove as follows

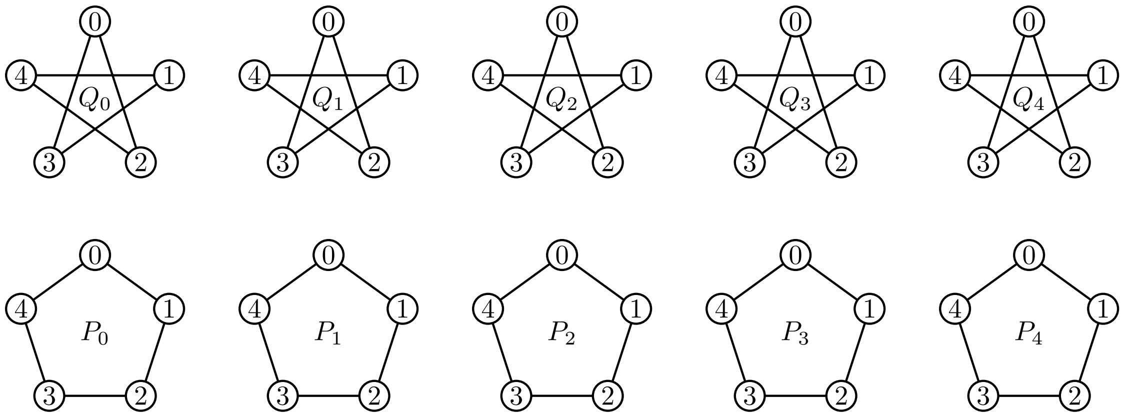
Theorem 1. *There is a unique* $(7,\ 2)$*-Moore graph up to isomorphism.*

Theorem 2. *The Hoffman-Singleton graph is vertex transitive, moreover all automorphisms are uniquely characterised by their action on a 5-cycle and a vertex at distance 2 from all points in that 5-cycle.*

1 Pentagons and Pentagrams Construction

We will begin by showing the existence of a $(2,\ 7)$-Moore graph through a direct construction. The essential character of our construction will reappear in our proof of uniqueness, hence we present it here to both build our intuition and prevent a break in subsequent exposition. The construction we present is the pentagons and pentagrams construction of the Hoffman-Singleton graph.

Define the graph G such that $V(G) = P \cup Q$, $P = \bigcup P_i$, $Q = \bigcup Q_i$, $P_i = \{p_{i,j}\}$ and $Q_i = \{q_{i,j}\}$ for $0 \leq i, j \leq 4$. Informally, each P_i is a pentagon, and each Q_i is a pentagram. The edges of G are given by the relations: $p_{i,j} \sim p_{i,j\pm1}$; $q_{i,j} \sim q_{i,j\pm2}$; $p_{i,j} \sim q_{k,ik+j}$; and $q_{i,j} \sim p_{k,ik+j}$, for $0 \leq i, j, k \leq 4$, and all arithmetic done modulo 5. We draw a diagram for clarity, noting that the non-trivial edges connecting vertices of P and Q are omitted.



Lemma 1. *The graph* G *is diameter 2.*

Proof. We will prove this by directly demonstrating paths of length 2 or less from an arbitrary $p_{i,j}$ to any other vertex, and noting that the symmetric argument holds for an arbitrary $q_{i,j}$.

Destination	Distance	Path	Amount
$p_{i,j}$	0	-	1
$p_{i,j\pm1}$	1	-	2
$q_{k,ik+j}$	1	-	5
$p_{i,j\pm2}$	2	$p_{i,j\pm1}$	2
$q_{k,ik+j\pm1}$	10	$p_{i,j\pm1}$	10
$q_{k,ik+j\pm2}$	10	$q_{k,ik+j}$	10
$p_{n,n(i+k)+j}$	20	$q_{k,ik+j}$, $(i \neq n)$	20

We now see that $p_{i,j}$ is within distance 2 of 50 distinct vertices of G. As $|V(G)| = 50$, $p_{i,j}$ was arbitrary, and the argument is symmteric for each $q_{i,j}$, we conclude that G is diameter 2. ☐

Corollary 1. *The graph* G *is girth 5.*

Proof. Let $v \in V(G)$ be an arbitrary vertex in G, and consider a breadth first tree $T < G$ rooted at v. As G is diameter 2, we have that all vertices in T are within distance 2 of v. As G is degree 7, T contains at most $1 + 7 + 7 \times 6 = 50$ vertices, which happens if and only if all paths of length 1 or 2 from v lead to distinct vertices. As $|V(G)| = 50$ we conclude that this happens, and conclude that v is in no cycles of length 4 or less. Hence, G is girth at least 5. Clearly G cannot be girth 6 or more as then there would be two vertices distance 3 or more apart. Hence, G is girth exactly 5. ☐

Hence we see that the graph G is diameter 2 and girth 5, and so is a Moore graph. As it is degree 7, it is a (7, 2)-Moore graph.

2 Proof of Uniqueness

For completeness, we begin with a simple lemma counting the 5-cycles in a $(7,2)$-Moore graph.

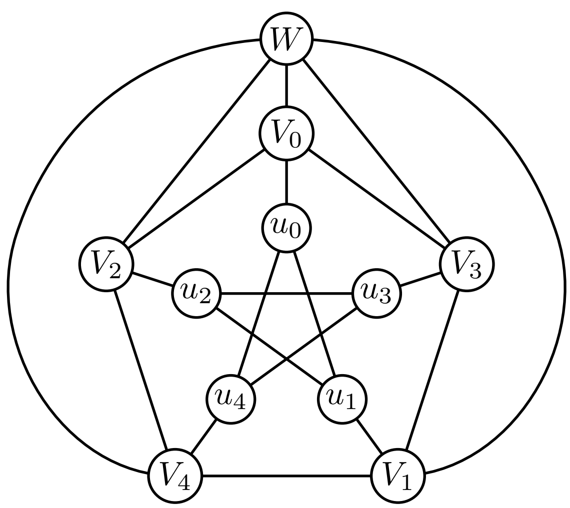
Lemma 2. *A $(7,2)$-Moore graph contains 1,260 5-cycles.*

Proof. Let G be a $(7,2)$-Moore graph, and let $u \in \mathrm{V}(G)$. Consider a breadth first search tree T from u in G. As G is diameter 2, every vertex in G appears in T at either distance 1 or 2 from u. As G is regular of degree 7, there are 7 vertices at distance 1 from u, and as the girth of G is 5, from each of these vertices there are 6 distinct vertices at distance 2 from u. Our tree contains 50 vertices, and thus 49 edges, and by regularity of G we have that G contains 175 edges, hence there are 126 edges in $A = \mathrm{E}(T) \setminus \mathrm{E}(V)$. We note that each edge in A must connect two vertices at distance 2 from u, and must correspond to a 5-cycle going through u. Conversely, each 5-cycle through u corresponds to an edge in A, and so u is in $|A| = 126$ 5-cycles. As u was an arbitrary vertex of G, we see that all vertices of G are in 126 5-cycles, and hence G contains $126 \times |\mathrm{V}(G)|/5 = 1,260$ distinct 5-cycles. □

Before we begin the proof of Theorem 1 we note that due to the frequent use of the diameter and girth constraints in deducing the structure of a $(7,2)$-Moore graph we will use the phrases *by girth* and *by diameter* as short hand. In addition, all numeric subscripts used will be considered to be modulo 5 in the obvious way.

We now prove Theorem 1.

Proof. Let G be a $(7,2)$-Moore graph, and $U = \{u_i\}$ be a 5-cycle in G. Each u_i must be adjacent to 5 vertices not in U, call these V_i. We must have $V_i \cap V_j = \emptyset$ by girth. Let $V = \bigcup_{i=0}^{4} V_i$, and $W = \mathrm{V}(G) \setminus (U \cup V)$. Considering a vertex $w \in W$ we have that w must have at least one neighbour in each V_i by diameter, and at most one neighbour in each V_i by girth. Now consider some $v \in V_i$. We have $v \not\sim v'$ for any $v' \in V_{i\pm1}$ by girth. By diameter, there is at least one $v' \in V_{i\pm2}$ such that $v \sim v'$, and by girth there is at most one such v'. We may draw a diagram of the situation as follows:



Now let $A = U \cup W$ and $B = V$. From what we have shown so far we see that the induced subgraphs of G from the vertices of A and B are both regular of degree 2. We now count the 5-cycles of G.

Let $E_A \subseteq \mathrm{E}(G)$ be the set of edges of G with both end points in A, $E_B \subseteq \mathrm{E}(G)$ be defined similarly, and $E_C = \mathrm{E}(G) \setminus (E_A \cup E_B)$. We now count the number of 5-cycles in G by considering in which sets their edges lie.

- No 5-cycles have all of their edges in E_C, as vertices in any cycle contained in E_C alternate between being in A and B, hence any such cycle is even length.

- A 5-cycle which has exactly one edge in E_A must have either two edges in E_B or no edges in E_B.
- A 5-cycle which contains exactly two edges in E_B must have both edges adjacent and exactly one edge in E_A.
- A 5-cycle which has at least three edges in E_A either has all of its edges in E_A or two edges in E_C.
- These observations are symmetric with respect to E_A and E_B.

From these observations, we may count the 5-cycles in G by counting all of those with exactly one edge in either E_A or E_B, and all those with three or more edges in either E_A or E_B.

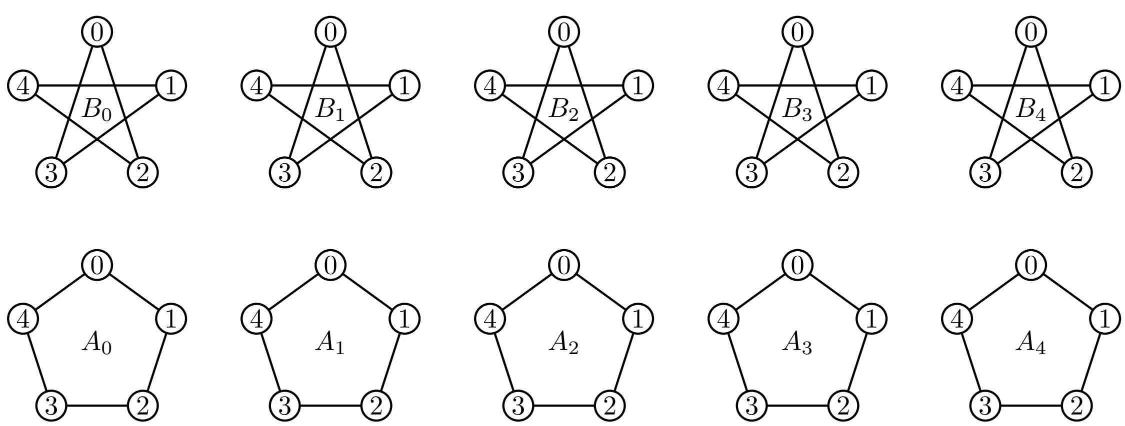
Given any edge in E_A, say $u \sim v$, there exist five vertices $u' \in B$ such that $u' \sim u$, and five vertices $v' \in B$ such that $v' \sim v$. Hence there are twenty five 3-paths of the form $u' \sim u \sim v \sim v'$. Each of these 3-paths is in a distinct 5-cycle, and clearly every 5-cycle containing $u \sim v$ corresponds to a distinct 3-path by girth. Hence each edge in E_A is contained in 25 5-cycles with exactly one edge in E_A. Hence there are $25 \times |E_A| = 625$ 5-cycles containing exactly one edge in E_A. By symmetry of E_A and E_B there are 1,250 5-cycles containing exactly one edge in either E_A or E_B.

To count the 5-cycles of G with three or more edges in E_A we consider the subgraph H of G induced by A. We have already shown that H is 2-regular, and so it is composed of disjoint cycles, which are of length at least 5 by girth. We now consider the lengths of the cycles in H.

- A 5-cycle in H corresponds to only one 5-cycle in G with 3 or more edges in E_A, and so each edge in this 5-cycle adds an average of $1/5$ of a 5-cycle to G.
- For $n \geq 6$, an n-cycle in H corresponds to n 5-cycles in G with 3 or more edges in E_A, and so each edge in this n-cycle corresponds to a 5-cycle in G.

Hence we see that the number of 5-cycles in G is at a minimum if and only if H is composed of disjoint 5-cycles. In this case there are five 5-cycles with 3 edges or more in E_A. By symmetry, we see the same for the subgraph of G induced by the vertices of B. In this case there are ten 5-cycles with exactly 3 edges in either E_A or E_B, which is exactly the number of unaccounted 5-cycles in G. Hence the subgraphs of G induced by the vertices of A and B are composed of 5 disjoint 5-cycles.

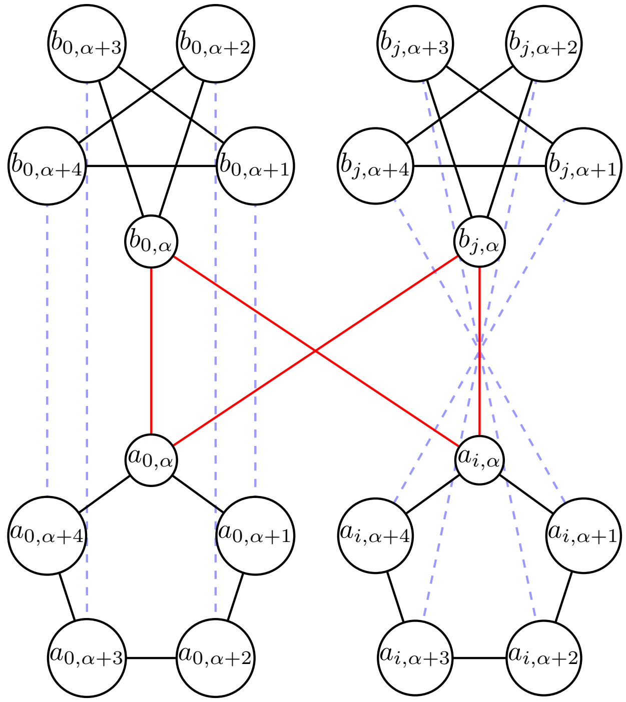
Now we have derived that G must be partitioned into two sets of 5-cycles with all remaining edges running between them. This can be drawn as the familiar pentagrams and pentagons as follows:



We now label the subgraph of G induced by A as A, the 5-cycles in A as $\{A_i\}$, the vertices of A_i as $\{a_{i,j}\}$, and the same labellings for B. We label such that $a_{i,j} \sim a_{i,j\pm1}$ and $b_{i,j} \sim b_{i,j\pm2}$. If we consider the beginning of our argument, choosing A_i as U, we see that for each B_j the subgraph induced by A_i and B_j must be a Petersen graph. Hence, there must exist some n such that either $a_{i,k} \sim b_{j,n+k}$ or $a_{i,k} \sim b_{j,n-k}$. We will say in the first case that A_i and B_j are *positively oriented*, and in the second case that they are *negatively oriented*.

Now we may label $\{A_i\}$ and $\{B_i\}$ such that each pair A_0, B_i and A_i, B_0 is positively oriented. We may also label in such a way that $a_{0,0} \sim b_{i,0}$ and $a_{i,0} \sim b_{0,0}$. Suppose there is some $1 \leq i, j \leq 4$ such that A_i and

B_j are negatively oriented. Hence there is some n such that $a_{i,k} \sim b_{j,n-k}$ for each k. We see the equation $k \equiv n - k \pmod 5$ has the solution $k \equiv 3n \pmod 5$. Hence letting $\alpha \equiv 3n \pmod 5$, we have the cycle $a_{0,\alpha} \sim b_{0,\alpha} \sim a_{i,\alpha} \sim b_{j,\alpha} \sim a_{0,\alpha}$, contradicting girth. We may draw a diagram of this situation as follows:



Hence we must have that all pairs A_i, B_j are positively oriented. Hence for each i, j there exists some n such that $a_{i,k} \sim b_{j,k+n}$, call this n the *offset* of A_i and B_j. We now define the matrix M to have its entry $m_{i,j}$ as the offest of A_i and B_j. We now note that any matrix $M \in \mathbb{M}_5(\mathbb{Z}_5)$ specifies all remaining edges of a candidate graph, which we may denote as G_M.

We now define an equivalence relation $\approx \subseteq \mathbb{M}_5(\mathbb{Z}_5) \times \mathbb{M}_5(\mathbb{Z}_5)$ with the aim of showing that if $M \approx N$ then $G_M \cong G_N$. Define $\approx \subseteq \mathbb{M}_5(\mathbb{Z}_5) \times \mathbb{M}_5(\mathbb{Z}_5)$ to be the transitive closure of the following:

1. $M \approx N$ for all $M, N \in \mathbb{M}_5(\mathbb{Z}_5)$ such that N can be formed by permuting the rows or columns of N;

2. $M \approx N$ for all $M, N \in \mathbb{M}_5(\mathbb{Z}_5)$ such that N can be formed by adding some $\alpha \in \mathbb{Z}_5$ to each entry in the first row or column of M.

To show that $G_M \cong G_N$ if $M \approx N$, we show that if only one of the above rules is required to show $M \approx N$, then $G_M \cong G_N$, the result then follows as the isomorphisms we demonstrate may be composed. In the following we only consider operations on the rows, noting that the case of the columns is symmetric.

In the case of rule 1, suppose $\pi \in S_5$ is the permutation of the rows necessary to transform M to N, then the mapping $\phi : G_M \to G_N$ given by $\phi(a_{i,j}) = a_{\pi(i),j}$ and $\phi(b_{i,j}) = b_{i,j}$ is an isomorphism from G_M to G_N.

In the case of rule 2, suppose that N is formed by adding α to each entry in the first row of M, then the mapping $\phi : G_M \to G_N$ given by $\phi(a_{0,j}) = a_{0,j+\alpha}$, and $\phi(v) = v$ for all other $v \in \mathrm{V}(G_M)$ is an isomorphism from G_M to G_N.

Now suppose that G_M is a $(2,7)$-Moore graph. We wish to show that for any $x, y, i, j \in \mathbb{Z}_5$ with $i \neq j$ and $x \neq y$ that $m_{x,i} - m_{y,i} \neq m_{x,j} - m_{y,j}$. We perform the following transformations on M:

1. permute the columns with some $\pi \in S_5$ such that $\pi(i) = 0$ and $\pi(j) = 1$;

2. permute the columns with some $\pi \in S_5$ such that $\pi(x) = 0$ and $\pi(y) = 1$;

3. add $-m_{x,i}$ to column 0 and $-m_{x,j}$ to column 1;

4. add $-(m_{y,i} - m_{x,i})$ to row 1.

$$M = \begin{pmatrix} - & - & - & - & - \\ - & m_{x,i} & - & m_{x,j} & - \\ - & - & - & - & - \\ - & m_{y,i} & - & m_{y,j} & - \\ - & - & - & - & - \end{pmatrix} \xrightarrow{1} \begin{pmatrix} - & - & - & - & - \\ m_{x,i} & m_{x,j} & - & - & - \\ - & - & - & - & - \\ m_{y,i} & m_{y,j} & - & - & - \\ - & - & - & - & - \end{pmatrix}$$

$$\xrightarrow{2} \begin{pmatrix} m_{x,i} & m_{x,j} & - & - & - \\ m_{y,i} & m_{y,j} & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 0 & 0 & - & - & - \\ m_{y,i} - m_{x,i} & m_{y,j} - m_{x,j} & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{pmatrix}$$

$$\xrightarrow{4} \begin{pmatrix} 0 & 0 & - & - & - \\ 0 & (m_{y,j} - m_{x,j}) - (m_{y,i} - m_{x,i}) & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{pmatrix} = N$$

Now we see that if $(m_{y,j}-m_{x,j})-(m_{y,i}-m_{x,i}) = 0$ then G_N contains the cycle $a_{0,0} \sim b_{0,0} \sim a_{1,0} \sim b_{1,0} \sim a_{0,0}$, and so is not a $(7,2)$-Moore graph, contradicting $G_N \cong G_M$. Therefore we must have $(m_{y,j}-m_{x,j})-(m_{y,i}-m_{x,i}) \neq 0$. We note that this observation is symmetric and the roles of columns and rows may be reversed. We shall refer to this result as *distinct differences*.

We now use this observation to find an isomorphism of a $(2,7)$-Moore graph G_M to a standard form. Let G_M be a $(2,7)$-Moore graph. We perform the following transformations on M to arrive at our standard form:

1. add $-a_i$ to each entry in column i and $-b_i$ to each entry in row $(i+1)$ so that the first row and column of M become all zeroes. We note that distinct differences now implies every row and column contains each value 0 to 4 exactly once;
2. the entries c_i are the numbers 1 through 4, we permute them to be in order;
3. the entries d_i are the numbers 2 through 4, we permute them to be in order.

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & - & - & - & - \\ b_2 & - & - & - & - \\ b_3 & - & - & - & - \\ b_4 & - & - & - & - \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & c_1 & c_2 & c_3 & c_4 \\ 0 & - & - & - & - \\ 0 & - & - & - & - \\ 0 & - & - & - & - \end{pmatrix}$$

$$\xrightarrow{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & d_1 & - & - & - \\ 0 & d_2 & - & - & - \\ 0 & d_3 & - & - & - \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & e_{1,1} & e_{1,2} & e_{1,3} \\ 0 & 3 & e_{2,1} & e_{2,2} & e_{2,3} \\ 0 & 4 & e_{3,1} & e_{3,2} & e_{3,3} \end{pmatrix} = N.$$

Finally, we examine the matrix N. Distinct differences tells us that $n_{3,4} = e_{1,2}$ may not be 0, 2, 3 or 4 (not 4 as $n_{2,3} - n_{3,3} = 1 \Rightarrow n_{3,4} \neq n_{2,4} + 1 = 4$); hence $e_{1,2} = 1$ and by symmetry $e_{2,1} = 1$. Continuing in this fashion we deduce in order $e_{1,1} = 4$, $e_{1,3} = e_{3,1} = 3$, $e_{2,3} = e_{3,2} = 2$, $e_{2,2} = 4$ and $e_{3,3} = 1$. Hence we have shown that if G_M is a $(2,7)$-Moore graph then $G_M \cong G_N$ where

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 1 & 3 \\ 0 & 3 & 1 & 4 & 2 \\ 0 & 4 & 3 & 2 & 1 \end{pmatrix}.$$

Hence, we see that if a (7, 2)-Moore graph exists, then it is unique to isomorphism. As we have already demonstrated such a graph through construction, we conclude that a (7, 2)-Moore graph exists and is unique to isomporhism. □

3 Automorphism Group

We now further pursue our argument to characterise the automorphisms of the Hoffman-Singleton graph, and show that the Hoffman-Singleton graph is vertex transitive. First we introduce a lemma.

Lemma 3. *The only automorphism of G fixing A_0 and $a_{1,0}$ is the identity.*

Proof. Let ϕ be an automorphism of G fixing A_0 and $a_{1,0}$. The paths of length 2 from each $a_{0,i}$ to $a_{1,0}$ are unique due to girth, and are given as $a_{0,i} \sim b_{i,i} \sim a_{1,0}$. Hence ϕ also fixes $b_{i,i}$. In addition, ϕ must also fix the orientation of each cycle B_i, and hence as it fixes a single vertex of each B_i and the orientation of each cycle B_i, all vertices in B are fixed by ϕ. As we now have ϕ fixes B_0 and $b_{1,0}$, a symmetric argument shows that ϕ fixes A, and hence ϕ fixes all of $\mathrm{V}(G)$ and so is the identity. □

We now prove Theorem 2.

Proof. Let G_N be the Hoffman-Singleton graph with N in the standard form as before, let G be the Hoffman-Singleton graph, and let $\psi : G_N \to G$ be an isomorphism. Consider any 5-cycle $U = \{u_i\}$ of G. Adopting the previous notation, label $A_0 = U$ so that $a_{0,i} = u_i$. Now pick a vertex at distance two from each vertex in U, say v. We know that v cannot be in B, as each vertex in B is at distance 1 from one vertex in A_0. Hence v is some vertex in $A' = \bigcup_{1\le j\le 4} A_j$. As each vertex in A' cannot be distance 1 from any vertex in A_0, and G is diameter 2, we see that each vertex in A' is distance 2 from all vertices in A_0, and so v could be any one of the vertices in A'. Now with U labelled as A_0 and v labelled as $a_{1,0}$ finish labelling G and derive the matrix M such that $G = G_M$ in this labelling. We now perform the following transformations on M, noting that each associated isomorphism maps A_0 to A_0 and $a_{1,0}$ to $a_{1,0}$ in the corresponding graphs:

1. add $-a_i$ to column i;
2. permute the columns so that each $b_i = i - 1$;
3. add $-c_i$ to row $(i + 1)$;
4. permute the rows so that each $d_i = i + 1$.

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{pmatrix} \xrightarrow{1} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{pmatrix}$$

$$\xrightarrow{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ c_1 & - & - & - & - \\ c_2 & - & - & - & - \\ c_3 & - & - & - & - \end{pmatrix} \xrightarrow{3} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & d_1 & - & - & - \\ 0 & d_2 & - & - & - \\ 0 & d_3 & - & - & - \end{pmatrix}$$

$$\xrightarrow{4} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & - & - & - \\ 0 & 3 & - & - & - \\ 0 & 4 & - & - & - \end{pmatrix} = M'.$$

From our previous work we immediately deduce that $M' = N$, and so we have an automorphism $\psi_{U,v} : G \to G_N$ such that $\psi_{U,v}(u_i) = a_{0,i}$ and $\psi_{U,v}(v) = a_{1,0}$. Now let $\theta_{U,v} : G_N \to G_N$ be given by $\theta_{U,v} = \psi_{U,v} \circ \phi$. As U and v were arbitrary, there exists some $\theta_{U,v}$ an automorphism of G_N which maps any arbitrary 5-cycle of G_N to A_0 and any arbitrary vertex at distance 2 from all vertices in that 5-cycle to $a_{1,0}$.

We have now shown that an automorphism ϕ exists which maps any 5-cycle U in G_N to A_0 and a vertex v at distance 2 from each vertex in U to $a_{1,0}$. Let ϕ, ϕ' be two automorphisms with this property, and form the automorphism $\psi = \phi' \circ \phi^{-1}$. We have that ψ fixes A_0 and $a_{1,0}$, hence by Lemma 3 we have that ψ is the identity, and so $\phi = \phi'$. Hence there is exactly one automorphism ϕ which maps U to A_0 and v to $a_{1,0}$. □

4 Conclusion

At this point it is natural to ask what more we may prove using this approach. However, with reference to other approaches, such as that in [4], it seems very likely that any further extension of the approach used above would be laborious for the purpose of developing further insight into the structure of the automorphism group of the Hoffman-Singleton graph.

We also now note that the reason the idea of counting the 5-cycles in the Hoffman-Singleton graph in two ways allows us to deduce its structure is likely due to the fact the Hoffman-Singleton graph achieves the theoretical upper bound of the number of 5-cycles given in [7]. Hence it may be worth investigating graphs which have a maximal number of convex cycles in order to further explore the topic of Moore graphs, especially as small examples of such graphs can be constructed to help build insight into the area.

In addition, when one attempts to follow this approach in the case of a potential (57, 2)-Moore graph graph, one finds that one cannot replicate the bipartition into 5-cycles, as in the initial stage the sets $A = U \cup W$ and $B = V$ are of hugely different sizes (2,975 and 275 respectively), and further the vertices in the set W must be adjacent to exactly 5 vertices in V, and hence 7 is the only possible valency so that the subgraph induced by the vertices of W is 2-regular. Should other graphs be found containing an optimal number of convex 5-cycles for their size which also are a bipartition of 5-cycles, like the Petersen and Hoffman-Singleton graphs, this may give an insight into a possible approach to answerting the question of a potential (57, 2)-Moore graph.

5 Acknowledgements

The author would like to thank Jozef Širáň for his help and advice with preparing this manuscript.

References

[1] Chris Godsil and Gordon Royle, Algebraic Graph Theory *Springer*

[2] Robert R. Singleton, Alan J. Hoffman, On Moore Graphs with Diameters 2 and 3, *IBM Journal of Research and Development*, 1960, 497-504.

[3] E Bannai, Tsatsumi Ito, On Finite Moore Graphs, *Journal of the Faculty of Science of the University of Tokyo*, 1973, 191-208.

[4] Paul R Hafner, The Hoffman-Singleton Graph and its Automorphisms, *Journal of Algebraic Combinatorics*, 2002, 7-12.

[5] R. M. Damerell, On Moore Graphs, *Mathematical Proceedings of the Cambridge Philosophical Society*, 1973, 227-236.

[6] Peter J. Cameron, Permutation Groups, *Cambridge University Press*, 1999.

[7] Jernej Azarija, Sandi Klavžar, Moore Graphs and Cycles are Extremal Graphs for Convex Cycles, *Journal of Graph Theory*, 2014, 34-42.

[8] L. O. James, A Combinatorial Proof that the Moore (7, 2) Graph is Unique, *Utilitas Mathematica, Vol. 5*, 1974, 79-84.